

---

# IP-DIRICHLET MEASURES AND IP-RIGID DYNAMICAL SYSTEMS: AN APPROACH VIA GENERALIZED RIESZ PRODUCTS

by

Sophie Grivaux

---

**Abstract.** — If  $(n_k)_{k \geq 1}$  is a strictly increasing sequence of integers, a continuous probability measure  $\sigma$  on the unit circle  $\mathbb{T}$  is said to be IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$  if  $\hat{\sigma}(\sum_{k \in F} n_k) \rightarrow 1$  as  $F$  runs over all non-empty finite subsets  $F$  of  $\mathbb{N}$  and the minimum of  $F$  tends to infinity. IP-Dirichlet measures and their connections with IP-rigid dynamical systems have been investigated recently by Aaronson, Hosseini and Lemańczyk. We simplify and generalize some of their results, using an approach involving generalized Riesz products.

## 1. Introduction

We will be interested in this paper in IP-Dirichlet probability measures on the unit circle  $\mathbb{T} = \{\lambda \in \mathbb{C} ; |\lambda| = 1\}$  with respect to a strictly increasing sequence  $(n_k)_{k \geq 1}$  of positive integers. Recall that a probability measure  $\mu$  on  $\mathbb{T}$  is said to be a *Dirichlet measure* when there exists a strictly increasing sequence  $(p_k)_{k \geq 1}$  of integers such that the monomials  $z^{p_k}$  tend to 1 on  $\mathbb{T}$  as  $k$  tends to infinity with respect to the norm of  $L^p(\mu)$ , where  $1 \leq p < +\infty$ . This is equivalent to requiring that the Fourier coefficients  $\hat{\mu}(p_k)$  of the measure  $\mu$  tend to 1 as  $k$  tends to infinity. If  $(n_k)_{k \geq 1}$  is a (fixed) strictly increasing sequence of integers, we say that  $\mu$  is a *Dirichlet measure with respect to the sequence*  $(n_k)_{k \geq 1}$  if  $\hat{\mu}(n_k) \rightarrow 1$  as  $k \rightarrow +\infty$ . Let  $\mathcal{F}$  denote the set of all non-empty finite subsets of  $\mathbb{N}$ . The measure  $\mu$  is said to be *IP-Dirichlet with respect to the sequence*  $(n_k)_{k \geq 1}$  if

$$\hat{\mu}\left(\sum_{k \in F} n_k\right) \rightarrow 1 \quad \text{as } \min(F) \rightarrow +\infty, \quad F \in \mathcal{F}.$$

In other words: for all  $\varepsilon > 0$  there exists a  $k_0 \geq 0$  such that whenever  $F$  is a finite subset of  $\{k_0, k_0 + 1, \dots\}$ ,

$$|\hat{\mu}\left(\sum_{k \in F} n_k\right) - 1| \geq \varepsilon.$$

Our starting point for this paper is the work [1] by Aaronson, Hosseini and Lemańczyk, where IP-Dirichlet measures are studied in connection with rigidity phenomena for dynamical systems. Let  $(X, \mathcal{B}, m)$  denote a standard non-atomic probability space and let

---

**2000 Mathematics Subject Classification.** — 37A25, 42A16, 42A55, 37A45.

**Key words and phrases.** — Dirichlet and IP-Dirichlet measures, rigid and IP-rigid weakly mixing dynamical systems, generalized Riesz products.

$T$  be a measure-preserving transformation of  $(X, \mathcal{B}, m)$ . Let again  $(n_k)_{k \geq 1}$  be a strictly increasing sequence of integers.

**Definition 1.1.** — The transformation  $T$  is said to be *rigid with respect to*  $(n_k)_{k \geq 1}$  if  $m(T^{-n_k} A \Delta A) \rightarrow 0$  as  $n_k \rightarrow +\infty$  for all sets  $A \in \mathcal{B}$ , or, equivalently, if for all functions  $f \in L^2(X, \mathcal{B}, m)$ ,  $\|f \circ T^{n_k} - f\|_{L^2(X, \mathcal{B}, m)} \rightarrow 0$  as  $k \rightarrow +\infty$ .

If we denote by  $\sigma_T$  the restricted spectral type of  $T$ , i.e. the spectral type of the Koopman operator  $U_T$  of  $T$  acting on functions of  $L^2(X, \mathcal{B}, m)$  of mean zero, then it is not difficult to see that  $T$  is rigid with respect to  $(n_k)_{k \geq 1}$  if and only if  $\sigma_T$  is a Dirichlet measure with respect to the sequence  $(n_k)_{k \geq 1}$ .

Rigidity phenomena for weakly mixing transformations have been investigated recently in the papers [3] and [4], where in particular the following question was considered: given a sequence  $(n_k)_{k \geq 1}$  of integers, when is it true that there exists a weakly mixing transformation  $T$  of some probability space  $(X, \mathcal{B}, m)$  which is rigid with respect to  $(n_k)_{k \geq 1}$ ? When this is true, we say that  $(n_k)_{k \geq 1}$  is a *rigidity sequence*. It was proved in [3] and [4] that  $(n_k)_{k \geq 1}$  is a rigidity sequence if and only if there exists a continuous probability measure  $\sigma$  on  $\mathbb{T}$  which is Dirichlet with respect to  $(n_k)_{k \geq 1}$ .

It is then natural to consider IP-rigidity for (weakly mixing) dynamical systems. This study was initiated in [3] and continued in [1].

**Definition 1.2.** — The system  $(X, \mathcal{B}, m; T)$  is said to be *IP-rigid with respect to the sequence*  $(n_k)_{k \geq 1}$  if for every  $A \in \mathcal{B}$ ,

$$m(T^{\sum_{k \in F} n_k} A \Delta A) \rightarrow 0 \quad \text{as } \min(F) \rightarrow +\infty, \quad F \in \mathcal{F}.$$

Just as with the notion of rigidity,  $T$  is IP-rigid with respect to  $(n_k)_{k \geq 1}$  if and only if  $\sigma_T$  is an IP-Dirichlet measure with respect to  $(n_k)_{k \geq 1}$ . Moreover, if we say that  $(n_k)_{k \geq 1}$  is an *IP-rigidity sequence* when there exists a weakly mixing dynamical system  $(X, \mathcal{B}, m; T)$  which is IP-rigid with respect to  $(n_k)_{k \geq 1}$ , then IP-rigidity sequences can be characterized in a similar fashion as rigidity sequences ([1, Prop. 1.2]):  $(n_k)_{k \geq 1}$  is an IP-rigidity sequence if and only if there exists a continuous probability measure  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ .

IP-Dirichlet measures are studied in detail in the paper [1], and one of the important features which is highlighted there is the connection between the existence of a measure which is IP-Dirichlet with respect to a certain sequence  $(n_k)_{k \geq 1}$  of integers, and the properties of the subgroups  $G_p((n_k))$  of the unit circle associated to  $(n_k)_{k \geq 1}$ : for  $1 \leq p < +\infty$ ,

$$G_p((n_k)) = \left\{ \lambda \in \mathbb{T} ; \sum_{k \geq 1} |\lambda^{n_k} - 1|^p < +\infty \right\}$$

and for  $p = +\infty$

$$G_\infty((n_k)) = \left\{ \lambda \in \mathbb{T} ; |\lambda^{n_k} - 1| \rightarrow 0 \text{ as } k \rightarrow +\infty \right\}.$$

The main result of [1] runs as follows:

**Theorem 1.3.** — [1, Th. 2] *Let  $(n_k)_{k \geq 1}$  be a strictly increasing sequence of integers. If  $\mu$  is a probability measure on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ , then  $\mu(G_2((n_k))) = 1$ .*

The converse of Theorem 1.3 is false [1, Ex. 4.2], as one can construct a sequence  $(n_k)_{k \geq 1}$  and a probability measure  $\mu$  on  $\mathbb{T}$  which is continuous, supported on  $G_2((n_k))$  (which is uncountable), and not IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . On the other hand, if  $\mu$  is a continuous probability measure such that  $\mu(G_1((n_k))) = 1$ , then  $\mu$  is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$  [1, Prop. 1]. Again, this is not a necessary and sufficient condition for being IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$  [1]: if  $(n_k)_{k \geq 1}$  is the sequence of integers defined by  $n_1 = 1$  and  $n_{k+1} = kn_k + 1$  for each  $k \geq 1$ , then there exists a continuous probability measure  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ , although  $G_1((n_k)) = \{1\}$ . Numerous examples of sequences  $(n_k)_{k \geq 1}$  with respect to which there exist IP-Dirichlet continuous probability measures are given in [1] as well. For instance, such sequences are characterized among sequences  $(n_k)_{k \geq 1}$  such that  $n_k$  divides  $n_{k+1}$  for each  $k$ , and among sequences which are denominators of the best rational approximants  $\frac{p_k}{q_k}$  of an irrational number  $\alpha \in (0, 1)$ , obtained via the continued fraction expansion. It is also proved in [1] that sequences  $(n_k)_{k \geq 1}$  such that the series  $\sum_{k \geq 1} (n_k/n_{k+1})^2$  is convergent admit a continuous IP-Dirichlet probability measure.

Our aim in this paper is to simplify and generalize some of the results and examples of [1]. We first present an alternative proof of Theorem 1.3 above, which is completely elementary and much simpler than the proof of [1] which involves Mackey ranges over the dyadic adding machine. We then present a rather general way to construct IP-Dirichlet measures via generalized Riesz products. The argument which we use is inspired by results from [9] and [7, Section 4.2], where generalized Riesz products concentrated on some  $H_2$ -subgroups of the unit circle are constructed. Proposition 3.1 gives a bound from below on the Fourier coefficients of these Riesz products, and this enables us to obtain in Proposition 4.1 a sufficient condition on sets  $\{n_k\}$  of the form

$$(1) \quad \{n_k\} = \bigcup_{k \geq 1} \{p_k, q_{1,k} p_k, \dots, q_{r_k,k} p_k\},$$

where the  $q_{j,k}$ ,  $j = 1, \dots, r_k$ , are positive integers and the sequence  $(p_k)_{k \geq 1}$  is such that  $p_{k+1} > q_{r_k,k} p_k$  for each  $k \geq 1$ , for the existence of an associated continuous generalized Riesz product which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . This condition is best possible (Proposition 4.2). As a consequence of Proposition 4.1, we retrieve and improve a result of [1] which runs as follows: if  $(n_k)_{k \geq 1}$  is such that there exists an infinite subset  $S$  of  $\mathbb{N}$  such that

$$\sum_{k \in S} \frac{n_k}{n_{k+1}} < +\infty \quad \text{and} \quad n_k | n_{k+1} \quad \text{for each } k \notin S,$$

then there exists a continuous probability measure  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . This result is proved in [1] by constructing a rank-one weakly mixing system which is IP-rigid with respect to  $(n_k)_{k \geq 1}$ . Here we get a “dynamical system-free” proof of this statement, where the condition  $\sum_{k \in S} (n_k/n_{k+1}) < +\infty$  is replaced by the weaker condition  $\sum_{k \in S} (n_k/n_{k+1})^2 < +\infty$ .

**Theorem 1.4.** — *Let  $(n_k)_{k \geq 1}$  be a strictly increasing sequence of integers for which there exists an infinite subset  $S$  of  $\mathbb{N}$  such that*

$$\sum_{k \geq 1} \left( \frac{n_k}{n_{k+1}} \right)^2 < +\infty \quad \text{and } n_k | n_{k+1} \text{ for each } k \notin S.$$

*Then there exists a continuous generalized Riesz product  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ .*

Using again sets of the form (1), we then show that the converse of Theorem 1.3 is false in the strongest possible sense, thus strengthening Example 4.2 of [1]:

**Theorem 1.5.** — *There exists a strictly increasing sequence  $(n_k)_{k \geq 1}$  of integers such that  $G_2((n_k))$  is uncountable, but no continuous probability measure is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ .*

The last section of the paper gathers some observations concerning the Erdős-Taylor sequence  $(n_k)_{k \geq 1}$  defined by  $n_1 = 1$  and  $n_{k+1} = kn_k + 1$ , which is of interest in this context.

**Notation:** In the whole paper, we will denote by  $\{x\}$  the distance of the real number  $x$  to the nearest integer, by  $\lfloor x \rfloor$  the integer which is closest to  $x$  (if there are two such integers, we take the smallest one), and by  $\langle x \rangle$  the quantity  $x - \lfloor x \rfloor$ . Lastly, we denote by  $[x]$  the integer part of  $x$ .

## 2. An alternative proof of Theorem 1.3

Let  $(n_k)_{k \geq 1}$  be a strictly increasing sequence of integers. Suppose that the measure  $\mu$  on  $\mathbb{T}$  is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . For every  $\varepsilon > 0$  there exists an integer  $k_0$  such that for all sets  $F \in \mathcal{F}$  with  $\min(F) \geq k_0$ ,  $|\hat{\mu}(\sum_{k \in F} n_k) - 1| \leq \varepsilon$ . For every integer  $N \geq k_0$ , consider the quantities

$$\prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) = 2^{-(N-k_0+1)} \sum_{F \subseteq \{k_0, \dots, N\}} \lambda^{\sum_{k \in F} n_k}.$$

The notation on the righthand side of this display means that the sum is taken over all (possibly empty) finite subsets  $F$  of  $\{k_0, \dots, N\}$ . Integrating with respect to  $\mu$  yields that

$$\int_{\mathbb{T}} \prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) d\mu(\lambda) = 2^{-(N-k_0+1)} \sum_{F \subseteq \{k_0, \dots, N\}} \hat{\mu}(\sum_{k \in F} n_k),$$

so that

$$(2) \quad \left| \int_{\mathbb{T}} \prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) d\mu(\lambda) - 1 \right| \leq 2^{-(N-k_0+1)} \sum_{F \subseteq \{k_0, \dots, N\}} \left| \hat{\mu}(\sum_{k \in F} n_k) - 1 \right| \leq \varepsilon.$$

Let now

$$C = \{ \lambda \in \mathbb{T} \ ; \text{ the infinite product } \prod_{k=1}^{+\infty} \frac{1}{2} |1 + \lambda^{n_k}| \text{ converges to a non-zero limit} \}.$$

Observe that the set  $C$  does not depend on  $\varepsilon$  nor on  $k_0$ . For every  $\lambda \in \mathbb{T} \setminus C$ , the quantity  $\prod_{k=k_0}^N \frac{1}{2} |1 + \lambda^{n_k}|$  tends to 0 as  $N \rightarrow +\infty$ , and so by the dominated convergence theorem we get that

$$\int_{\mathbb{T} \setminus C} \prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) d\mu(\lambda) \rightarrow 0 \quad \text{as } N \rightarrow +\infty.$$

It then follows from (2) that

$$\left| \limsup_{N \rightarrow +\infty} \int_C \prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) d\mu(\lambda) - 1 \right| \leq \varepsilon$$

so that

$$\limsup_{N \rightarrow +\infty} \left| \int_C \prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) d\mu(\lambda) \right| \geq 1 - \varepsilon.$$

But

$$\left| \int_C \prod_{k=k_0}^N \frac{1}{2} (1 + \lambda^{n_k}) d\mu(\lambda) \right| \leq \mu(C),$$

hence  $\mu(C) \geq 1 - \varepsilon$ . This being true for any choice of  $\varepsilon$  in  $(0, 1)$ ,  $\mu(C) = 1$ , and so the product  $\prod_{k \geq 1} \frac{1}{2} |1 + \lambda^{n_k}|$  converges to a non-zero limit almost everywhere with respect to the measure  $\mu$ . If we now write elements  $\lambda \in C$  as  $\lambda = e^{2i\pi\theta}$ ,  $\theta \in [0, 1)$ , we have

$$\prod_{k \geq 1} \frac{1}{2} |1 + \lambda^{n_k}| = \prod_{k \geq 1} |\cos(\pi\theta n_k)|.$$

Since  $0 < |\cos(\pi\theta n_k)| \leq 1$  for all  $k \geq 1$ , this means that the series  $\sum_{k \geq 1} 1 - |\cos(\pi\theta n_k)|$  is convergent. In particular  $\{\theta n_k\} \rightarrow 0$  as  $k \rightarrow +\infty$ . As the quantities  $1 - |\cos(\pi\theta n_k)|$  and  $\frac{\pi^2}{2} \{\theta n_k\}^2$  are equivalent as  $k \rightarrow +\infty$ , we obtain that the series  $\sum_{k \geq 1} \{\theta n_k\}^2$  is convergent. But

$$|1 - \lambda^{n_k}|^2 = |1 - e^{2i\pi\theta n_k}|^2 \leq 4\pi^2 \{\theta n_k\}^2,$$

and it follows from this that the series  $\sum_{k \geq 1} |1 - \lambda^{n_k}|^2$  is convergent as soon as  $\lambda$  belongs to  $C$ . This proves our claim.

### 3. IP-Dirichlet generalized Riesz products

Our aim is now to give conditions on the sequence  $(n_k)_{k \geq 1}$  which imply the existence of a generalized Riesz product which is continuous and IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . For information about classical and generalized Riesz products, we refer for instance the reader to the papers [9] and [7] and to the books [6] and [10].

**Proposition 3.1.** — *Let  $(n_k)_{k \geq 1}$  be a strictly increasing sequence of integers. Suppose that there exists a strictly increasing sequence  $(m_k)_{k \geq 1}$  of integers such that*

$$(3) \quad n_{k+1} - 2 \sum_{j=1}^k m_j n_j \geq 1 \quad \text{for each } k \geq 1,$$

and

$$(4) \quad n_{k+1} - 2 \sum_{j=1}^k m_j n_j \longrightarrow +\infty \quad \text{as } k \longrightarrow +\infty.$$

For each  $k \geq 1$ , let  $p_k \geq 1$  be an integer such that  $p_k \pi \leq m_k + 2$ . There exists a continuous generalized Riesz product  $\sigma$  on  $\mathbb{T}$  such that for every finite subset  $F \in \mathcal{F}$  and every integers  $j_k$  in  $\{1, \dots, p_k\}$ ,  $k \in F$ , one has

$$(5) \quad \hat{\sigma} \left( \sum_{k \in F} j_k n_k \right) \geq \prod_{k \in F} \left( 1 - 3\pi^2 \left( \frac{p_k}{m_k + 2} \right)^2 \right)$$

and

$$(6) \quad \hat{\sigma} \left( \sum_{k \in F} n_k \right) = \prod_{k \in F} \cos \left( \frac{\pi}{m_k + 2} \right).$$

*Proof.* — For any integer  $k \geq 1$ , consider the polynomial  $P_k$  defined by

$$P_k(e^{2i\pi t}) = \frac{2}{m_k + 2} \left| \sum_{j=1}^{m_k+1} \sin \left( \frac{j\pi}{m_k + 2} \right) e^{2i\pi j t} \right|^2, \quad t \in [0, 1].$$

Each  $P_k$  is a nonnegative trigonometric polynomial. Its spectrum is the set  $\{-m_k, \dots, m_k\}$  and a straightforward computation shows that  $\hat{P}_k(0) = 1$ . Condition (3), which is a dissociation condition, implies that the probability measures  $\prod_{k=1}^N P_k(e^{2i\pi n_k t}) d\lambda(t)$  (where  $\lambda$  denotes here the normalized Lebesgue measure on  $\mathbb{T}$ ) converge in the  $w^*$  topology as  $N \rightarrow +\infty$  to a probability measure  $\sigma$  on  $\mathbb{T}$ , and that for each  $F \in \mathcal{F}$  and each integers  $j_k \in \{-m_k, \dots, m_k\}$ ,  $k \in F$ ,

$$\hat{\sigma} \left( \sum_{k \in F} j_k n_k \right) = \prod_{k \in F} \hat{P}_k(j_k),$$

while  $\hat{\sigma}(n) = 0$  when  $n$  is not of this form. In particular

$$\hat{\sigma} \left( \sum_{k \in F} n_k \right) = \prod_{k \in F} \hat{P}_k(1).$$

Before getting into precise computation of these Fourier coefficients, let us prove that  $\sigma$  is a continuous measure: this follows from condition (4). If

$$\sum_{j=1}^k m_j n_j < n < n_{k+1} - \sum_{j=1}^k m_j n_j,$$

then  $\hat{\sigma}(n) = 0$ . So the Fourier transform of  $\sigma$  vanishes on successive intervals  $I_k$  of length  $l_k = n_{k+1} - 2 \sum_{j=1}^k m_j n_j - 1$ . Since  $l_k$  tends to infinity with  $k$  by (4), it follows from the Wiener theorem that  $\sigma$  is continuous.

Let us now go back to the computation of the Fourier coefficients  $\hat{\sigma} \left( \sum_{k \in F} j_k n_k \right)$ . For each  $p \in \{1, \dots, m_k\}$ , we have

$$(7) \quad \hat{P}_k(p) = \frac{2}{m_k + 2} \sum_{j=1}^{m_k+1-p} \sin \left( \frac{(j+p)\pi}{m_k + 2} \right) \sin \left( \frac{j\pi}{m_k + 2} \right).$$

Standard computations yield the following expression for  $\hat{P}_k(p)$ :

$$\begin{aligned}
(8) \quad \hat{P}_k(p) &= \frac{1}{m_k + 2} \left( (m_k + 2 - p) \cos\left(\frac{p\pi}{m_k + 2}\right) + \sin\left(\frac{p\pi}{m_k + 2}\right) \cdot \frac{\cos\left(\frac{\pi}{m_k + 2}\right)}{\sin\left(\frac{\pi}{m_k + 2}\right)} \right) \\
&= \frac{1}{m_k + 2} \left( (m_k + 2 - p) \cos\left(\frac{p\pi}{m_k + 2}\right) + \cos\left(\frac{(p-1)\pi}{m_k + 2}\right) \cdot \cos\left(\frac{\pi}{m_k + 2}\right) \right. \\
&\quad \left. + \sin\left(\frac{(p-1)\pi}{m_k + 2}\right) \cdot \frac{\cos^2\left(\frac{\pi}{m_k + 2}\right)}{\sin\left(\frac{\pi}{m_k + 2}\right)} \right) \\
&= \dots = \frac{1}{m_k + 2} \left( (m_k + 2 - p) \cos\left(\frac{p\pi}{m_k + 2}\right) \right. \\
&\quad \left. + \sum_{j=1}^p \cos\left(\frac{(p-j)\pi}{m_k + 2}\right) \cos^j\left(\frac{\pi}{m_k + 2}\right) \right).
\end{aligned}$$

Observe now that for every  $x \in [0, 1]$ ,  $\cos x \geq 1 - x^2 \geq 0$ . For each  $k \geq 1$ , let  $p_k \geq 1$  be an integer such that  $p_k \pi \leq m_k + 2$ , and let  $p$  belong to the set  $\{1, \dots, p_k\}$ . Then  $(p-j)\pi \leq m_k + 2$  for every  $j \in \{0, \dots, p-1\}$ . Thus

$$\cos\left(\frac{p\pi}{m_k + 2}\right) \geq 1 - \pi^2 \frac{p^2}{(m_k + 2)^2} \quad \text{and} \quad \cos\left(\frac{(p-j)\pi}{m_k + 2}\right) \geq 1 - \pi^2 \frac{(p-j)^2}{(m_k + 2)^2}.$$

Moreover,  $\cos^j x \geq (1 - x^2)^j \geq 1 - jx^2$  for every  $x \in [0, 1]$  and every  $j \geq 1$ , so that

$$\cos^j\left(\frac{\pi}{m_k + 2}\right) \geq 1 - \pi^2 \frac{j}{(m_k + 2)^2}.$$

Putting things together, we obtain the estimate

$$\begin{aligned}
\hat{P}_k(p) &\geq \frac{1}{m_k + 2} \left( (m_k + 2 - p) \left( 1 - \pi^2 \frac{p^2}{(m_k + 2)^2} \right) \right. \\
&\quad \left. + \sum_{j=1}^p \left( 1 - \pi^2 \frac{(p-j)^2}{(m_k + 2)^2} \right) \left( 1 - \pi^2 \frac{j}{(m_k + 2)^2} \right) \right).
\end{aligned}$$

Now, for every  $j \in \{1, \dots, p-1\}$ ,

$$\begin{aligned}
\left( 1 - \pi^2 \frac{(p-j)^2}{(m_k + 2)^2} \right) \left( 1 - \pi^2 \frac{j}{(m_k + 2)^2} \right) &= 1 - \pi^2 \frac{(p-j)^2 + j}{(m_k + 2)^2} + \pi^4 \frac{j(p-j)^2}{(m_k + 2)^2} \\
&\geq 1 - \pi^2 \frac{(p-j)^2 + j}{(m_k + 2)^2} \geq 1 - 2\pi^2 \frac{p^2}{(m_k + 2)^2}.
\end{aligned}$$

Summing over  $j$  and putting together terms, we eventually obtain that

$$\begin{aligned}
\hat{P}_k(p) &\geq \frac{1}{m_k + 2} \left( (m_k + 2 - p) \left( 1 - \pi^2 \frac{p^2}{(m_k + 2)^2} \right) + p - 2\pi^2 \frac{p^3}{(m_k + 2)^2} \right) \\
&\geq 1 - \frac{1}{m_k + 2} (m_k + 2 - p) \pi^2 \left( \frac{p}{m_k + 2} \right)^2 - 2\pi^2 \left( \frac{p}{m_k + 2} \right)^3,
\end{aligned}$$

i.e. that

$$\begin{aligned}\hat{P}_k(p) &\geq 1 - \pi^2 \left( \frac{p}{m_k + 2} \right)^2 - 2\pi^2 \left( \frac{p}{m_k + 2} \right)^3 \\ &\geq 1 - \pi^2 \left( \frac{p_k}{m_k + 2} \right)^2 - 2\pi^2 \left( \frac{p_k}{m_k + 2} \right)^3 \quad \text{for each } p \in \{1, \dots, p_k\} \\ &\geq 1 - 3\pi^2 \left( \frac{p_k}{m_k + 2} \right)^2.\end{aligned}$$

Assertion (5) follows directly from the fact that  $\hat{\sigma}(\sum_{k \in F} j_k n_k) = \prod_{k \in F} \hat{P}_k(j_k)$ . Assertion (6) is straightforward: the expression in the first line of the display (8) applied to  $p = 1$  yields that  $\hat{P}_k(1) = \cos(\pi/(m_k + 2))$ . This finishes the proof of Proposition 3.1.  $\square$

Proposition 3.1 may appear a bit technical at first sight, but it turns out to be quite easy to apply. As a first example, we use it to obtain another proof of a result of [1, Prop. 3.2]:

**Corollary 3.2.** — *Let  $(n_k)_{k \geq 1}$  be a strictly increasing sequence of integers such that the series  $\sum_{k \geq 1} (n_k/n_{k+1})^2$  is convergent. There exists a continuous generalized Riesz product  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ .*

*Proof.* — Without loss of generality we can assume that  $\sum_{k \geq 1} (n_k/n_{k+1})^2 < 1/36$ . Let  $(\varepsilon_k)_{k \geq 1}$  be a sequence of real numbers with  $0 < \varepsilon_k < 1/2$  for each  $k \geq 2$ , with  $\varepsilon_1 = 0$ , going to zero as  $k$  tends to infinity, and such that

$$\sum_{k \geq 1} \left( \frac{1}{\varepsilon_{k+1}} \frac{n_k}{n_{k+1}} \right)^2 < \frac{1}{9}.$$

Then  $\varepsilon_{k+1} n_{k+1}/n_k > 3 > 2 + \varepsilon_k$ , so that if we define  $m_k = [(\varepsilon_{k+1} n_{k+1} - \varepsilon_k n_k)/2n_k]$  for each  $k \geq 1$ , each  $m_k$  is a positive integer. Moreover

$$n_{k+1} - 2 \sum_{j=1}^k m_j n_j \geq n_{k+1} - (\varepsilon_{k+1} n_{k+1} - \varepsilon_1 n_1) = (1 - \varepsilon_{k+1}) n_{k+1}$$

which tends to infinity as  $k$  tends to infinity, and is always greater than 1 because  $\varepsilon_{k+1} < 1/2$  and  $n_{k+1} \geq 2$  for each  $k \geq 1$ . Proposition 3.1 applies with this choice of the sequence  $(m_k)_{k \geq 1}$  and yields a continuous generalized Riesz product  $\sigma$  which satisfies

$$\hat{\sigma}\left(\sum_{k \in F} n_k\right) = \prod_{k \in F} \cos\left(\frac{\pi}{m_k + 2}\right) \quad \text{for each } F \in \mathcal{F}.$$

Now  $m_k$  is equivalent as  $k$  tends to infinity to the quantity  $\varepsilon_{k+1} n_{k+1}/2n_k$ , so that the series  $\sum_{k \geq 1} 1/(m_k + 2)^2$  is convergent. Hence the infinite product  $\prod_{k \geq 1} \cos(\pi/(m_k + 2))$  is convergent. For any  $\varepsilon > 0$ , let  $k_0$  be such that  $\prod_{k \geq k_0} \cos(\pi/(m_k + 2)) \geq 1 - \varepsilon$ . If  $F \in \mathcal{F}$  is such that  $\min(F) \geq k_0$ ,

$$\hat{\sigma}\left(\sum_{k \in F} n_k\right) = \prod_{k \in F} \cos\left(\frac{\pi}{m_k + 2}\right) \geq \prod_{k \geq k_0} \cos\left(\frac{\pi}{m_k + 2}\right) \geq 1 - \varepsilon,$$

and this proves that  $\sigma$  is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ .  $\square$



#### 4. An application to a special class of sets $\{n_k\}$

Proposition 3.1 applies especially well to a particular class of sequences  $(n_k)_{k \geq 1}$ , which we now proceed to investigate.

**Proposition 4.1.** — *Let  $(p_l)_{l \geq 1}$  be a strictly increasing sequence of integers. For each  $l \geq 1$ , let  $(q_{j,l})_{j=0,\dots,r_l}$  be a strictly increasing finite sequence of integers with  $q_{0,l} = 1$ , and set  $q_l = q_{0,l} + q_{1,l} + \dots + q_{r_l,l}$ . Suppose that  $p_{l+1} > q_{r_l,l} p_l$  for each  $l \geq 1$ , and that the series*

$$\sum_{l \geq 1} \left( \frac{q_l p_l}{p_{l+1}} \right)^2$$

*is convergent. Let  $(n_k)_{k \geq 1}$  be the strictly increasing sequence defined by*

$$\{n_k\} = \bigcup_{l \geq 1} \{p_l, q_{1,l} p_l, \dots, q_{r_l,l} p_l\}$$

*There exists a continuous generalized Riesz product  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to the sequence  $(n_k)_{k \geq 1}$ .*

*Proof.* — As in the proof of Corollary 3.2, we can suppose that  $\sum_{k \geq 1} (q_l p_l / p_{l+1})^2 < 1/324$ , and consider a sequence  $(\varepsilon_l)_{l \geq 1}$  going to zero as  $l$  tends to infinity with  $\varepsilon_1 = 0$  and  $0 < \varepsilon_l < 1/2$  for each  $l \geq 2$ , such that

$$\sum_{l \geq 1} \left( \frac{1}{\varepsilon_{l+1}} \frac{q_l p_l}{p_{l+1}} \right)^2 < \frac{1}{81}.$$

The same argument as in the proof of Corollary 3.2 shows that for  $l \geq 1$  the integers  $m_l = [(\varepsilon_{l+1} p_{l+1} - \varepsilon_l p_l) / (2p_l)]$  are positive, and that assumptions (3) and (4) of Proposition 3.1 are satisfied. As  $m_l$  is equivalent as  $l$  tends to infinity to  $(\varepsilon_{l+1} p_{l+1}) / (2p_l)$ ,  $q_l / (m_l + 2)$  is equivalent to  $(2q_l p_l) / (\varepsilon_{l+1} p_{l+1})$ . Our assumption implies then that the series

$$(9) \quad \sum_{l \geq 1} \left( \frac{q_l}{m_l + 2} \right)^2$$

is convergent, and that  $q_l \pi \leq m_l + 2$  for each  $l \geq 1$ . Applying Proposition 3.1 to the sequence  $(p_l)_{l \geq 1}$ , we get a continuous generalized Riesz product  $\sigma$ , and the estimates (5) yield that

$$\hat{\sigma} \left( \sum_{l \in F} \left( \sum_{j \in G_l} q_{j,l} \right) p_l \right) \geq \prod_{l \in F} \left( 1 - 3\pi^2 \left( \frac{q_l}{m_l + 2} \right)^2 \right)$$

for each set  $F \in \mathcal{F}$  and each subsets  $G_l$  of  $\{0, \dots, r_l\}$ ,  $l \in F$ . In order to show that the measure  $\sigma$  is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ , it remains to observe that the product on the right-hand side is convergent by (9). We then conclude as in the proof of Corollary 3.2.  $\square$

The proof of Theorem 1.4 is now a straightforward corollary of Proposition 4.1. Recall that we wish to prove that if  $(n_k)_{k \geq 1}$  is a sequence of integers for which there exists an infinite subset  $S$  of  $\mathbb{N}$  such that

$$\sum_{k \in S} \left( \frac{n_k}{n_{k+1}} \right)^2 < +\infty \quad \text{and} \quad n_k | n_{k+1} \quad \text{for each } k \notin S,$$

then there exists a continuous generalized Riesz product  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ .

*Proof of Theorem 1.4.* — Let  $\Phi : \mathbb{N} \rightarrow \mathbb{N}$  be a strictly increasing function such that  $S = \{\Phi(l), l \geq 1\}$ . Set  $p_l = n_{\Phi(l)+1}$  for  $l \geq 1$  and write for each  $k \in \{\Phi(l) + 1, \dots, \Phi(l+1)\}$

$$n_k = s_{0,l} s_{1,l} \dots s_{k-(\Phi(l)+1),l} p_l,$$

with  $s_{0,l} = 1$  and  $s_{j,l} \geq 2$  for each  $j = 1, \dots, \Phi(l+1) - (\Phi(l) + 1)$ . With the notation of Proposition 4.1 we have  $r_l = \Phi(l+1) - (\Phi(l) + 1)$  and

$$q_{k-(\Phi(l)+1),l} = s_{0,l} s_{1,l} \dots s_{k-(\Phi(l)+1),l}$$

Hence  $q_l = q_{0,l} + \dots + q_{r_l,l} = s_{0,l} + s_{0,l} s_{1,l} + \dots + s_{0,l} s_{1,l} \dots s_{r_l,l}$ . We have

$$\begin{aligned} \frac{q_l}{s_{0,l} s_{1,l} \dots s_{r_l,l}} &= 1 + \frac{1}{s_{r_l,l}} + \frac{1}{s_{r_l-1,l} s_{r_l,l}} + \dots + \frac{1}{s_{2,l} \dots s_{r_l,l}} + \frac{1}{s_{1,l} \dots s_{r_l,l}} \\ &\leq 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{r_l}} \quad \text{since } s_{j,l} \geq 2 \text{ for each } j = 1, \dots, r_l \\ &\leq 2. \end{aligned}$$

This yields that  $q_l \leq 2s_{0,l} s_{1,l} \dots s_{r_l,l} = 2q_{r_l,l}$  for each  $l \geq 1$ . Our assumption that the series  $\sum_{k \in S} (n_k/n_{k+1})^2$  is convergent means that the series  $\sum_{l \geq 1} (q_{r_l,l} p_l/p_{l+1})^2$  is convergent. Hence the series  $\sum_{l \geq 1} (q_l p_l/p_{l+1})^2$  is convergent and the conclusion follows from Proposition 4.1.  $\square$

Our next result shows the optimality of the assumption of Proposition 4.1 that the series  $\sum_{l \geq 1} (q_l p_l/p_{l+1})^2$  is convergent.

**Proposition 4.2.** — *Let  $(\gamma_l)_{l \geq 1}$  be any sequence of positive real numbers, going to zero as  $l$  goes to infinity, such that the series  $\sum_{l \geq 1} \gamma_l^2$  is divergent, with  $0 < \gamma_l < 1$  for each  $l \geq 2$ . Let  $(r_l)_{l \geq 1}$  be a sequence of integers growing to infinity so slowly that the series  $\sum_{l \geq 1} \gamma_l^2/r_l$  is divergent, with  $r_l \geq 2$  for each  $l \geq 1$ . Define a sequence  $(p_l)_{l \geq 1}$  of integers by setting  $p_1 = 1$  and  $p_{l+1} = \lfloor r_l^2/\gamma_l \rfloor p_l + 1$ . For each  $l \geq 1$ , we have  $p_{l+1} > r_l p_l$ . Define a strictly increasing sequence  $(n_k)_{k \geq 1}$  of integers by setting*

$$\{n_k\} = \bigcup_{l \geq 1} \{p_l, 2p_l, \dots, r_l p_l\}.$$

*Then no continuous measure  $\sigma$  on the unit circle can be IP-Dirichlet with respect to the sequence  $(n_k)_{k \geq 1}$ .*

*Proof.* — We are going to show that  $G_2((n_k)) = \{1\}$ . It will then follow from Theorem 1.3 that no continuous probability measure on  $\mathbb{T}$  can be IP-Dirichlet with respect to the sequence  $(n_k)_{k \geq 1}$ . Suppose that  $\lambda \in \mathbb{T} \setminus \{1\}$  is such that

$$(10) \quad \sum_{k \geq 1} |\lambda^{n_k} - 1|^2 = \sum_{l \geq 1} \sum_{j=1}^{r_l} |\lambda^{j p_l} - 1|^2 < +\infty.$$

Let  $C$  be a positive constant such that for each  $\theta \in [0, 1]$ ,  $\frac{1}{C}\{\theta\} \geq |e^{2i\pi\theta} - 1| \geq C\{\theta\}$ . Writing  $\lambda$  as  $\lambda = e^{2i\pi\theta}$ ,  $\theta \in [0, 1)$ , we have that

$$(11) \quad |\lambda^{j p_l} - 1| \geq C\{j p_l \theta\} \quad \text{for each } l \geq 1 \text{ and } j = 1, \dots, r_l.$$

Now  $\{\theta p_l\} < 1/r_l$  for sufficiently large  $l$ . Else the set  $\{j\theta p_l, j = 1, \dots, r_l\}$  would form a  $\{\theta p_l\}$ -dense net of  $[0, 1]$ , and this would contradict the fact, implied by (10) and (11), that the quantity  $\sum_{j=1}^{r_l} \{j\theta p_l\}^2$  tends to zero as  $l$  tends to infinity. Hence, for sufficiently large  $l$ ,  $\{j\theta p_l\} = j\{\theta p_l\}$  for every  $j = 1, \dots, r_l$ , and thus the series  $\sum_{l \geq 1} \sum_{j=1}^{r_l} j^2 |\lambda^{p_l} - 1|^2$  is convergent. As  $r_l$  tends to infinity with  $l$ , this means that the series

$$(12) \quad \sum_{l \geq 1} r_l^3 |\lambda^{p_l} - 1|^2$$

is convergent.

Let now  $(\delta_l)_{l \geq 1}$  be a sequence of real numbers going to zero so slowly that the series  $\sum_{l \geq 1} \frac{1}{r_l} \gamma_l^2 \delta_l^2$  is divergent. Suppose that  $|\lambda^{p_l} - 1| < \frac{\gamma_l}{r_l^2} \delta_l$  for infinitely many  $l$ . Then,

$$|\lambda^{\lceil \frac{r_l^2}{\gamma_l} \rceil p_l} - 1| < \delta_l \quad \text{for all these } l,$$

and by definition of  $p_{l+1}$ ,  $|\lambda^{p_{l+1}} - \lambda| < \delta_l$ . Letting  $l$  tend to infinity along this set of integers, and remembering that  $|\lambda^{p_{l+1}} - 1| \rightarrow 0$  as  $l$  tends to infinity, we get that  $\lambda = 1$ , which is contrary to our assumption. Hence  $|\lambda^{p_l} - 1| \geq \frac{\gamma_l}{r_l^2} \delta_l$  for all integers  $l$  sufficiently large. Combining this with (12), this implies that the series

$$\sum_{l \geq 1} r_l^3 \frac{\gamma_l^2}{r_l^4} \delta_l^2 = \sum_{l \geq 1} \frac{1}{r_l} \gamma_l^2 \delta_l^2$$

is convergent, which is again a contradiction. So  $G_2((n_k)) = \{1\}$  and we are done.  $\square$

Consider the sets  $\{n_k\}$  given by Proposition 4.2. With the notation of Proposition 4.1,  $q_l$  is equivalent to  $r_l^2/2$  as  $k$  tends to infinity, and the series  $\sum_{l \geq 1} (q_l p_l / p_{l+1})^2$  is divergent because  $(q_l p_l / p_{l+1})^2$  is equivalent to  $\gamma_l^2/4$ . This shows the optimality of the condition given in Proposition 4.1.

Looking at the construction of Proposition 4.2 from a different angle yields an example of a sequence  $(n_k)_{k \geq 1}$  such that  $G_2((n_k))$  is uncountable, but still no continuous probability measure on  $\mathbb{T}$  can be IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . This is Theorem 1.5.

## 5. Proof of Theorem 1.5

Recall that we aim to construct a strictly increasing sequence  $(n_k)_{k \geq 1}$  of integers such that  $G_2((n_k))$  is uncountable, but no continuous probability measure on  $\mathbb{T}$  is IP-Dirichlet with respect to the sequence  $(n_k)_{k \geq 1}$ . This sequence  $(n_k)_{k \geq 1}$  will be of the kind considered in the previous section. Consider first the sequence  $(p_l)_{l \geq 1}$  defined by  $p_1 = 1$  and  $p_{l+1} = \frac{l^2(l^2+1)}{2} p_l$  for all  $l \geq 1$ . We then define the sequence  $(n_k)_{k \geq 1}$  by setting

$$\{n_k ; k \geq 1\} = \bigcup_{l \geq 2} \{p_l, 2p_l, \dots, l^2 p_l\}.$$

As  $l^2 p_l < p_{l+1}$  for all  $l \geq 2$ , the sets  $\{p_l, 2p_l, \dots, l^2 p_l\}$  are consecutive sets of integers. Let  $(M_l)_{l \geq 1}$  be the unique sequence of integers such that  $\{n_{M_{l-1}+1}, \dots, n_{M_l}\} =$

$\{p_l, 2p_l, \dots, l^2 p_l\}$  for each  $l \geq 2$ . We now know (see for instance [2] or [4] for a proof) that there exists a perfect uncountable subset  $K$  of  $\mathbb{T}$  such that

$$|\lambda^{p_l} - 1| \leq C \frac{p_l}{p_{l+1}} \quad \text{for all } \lambda \in K \text{ and } l \geq 2,$$

where  $C$  is a positive universal constant. Hence for  $\lambda \in K$ ,  $l \geq 2$  and  $j \in \{1, \dots, l^2\}$  we have

$$|\lambda^{jp_l} - 1| \leq C j \frac{p_l}{p_{l+1}} \leq 2C l^2 \frac{1}{l^4} = \frac{2C}{l^2}.$$

Thus

$$\sum_{j=1}^{l^2} |\lambda^{jp_l} - 1|^2 \leq l^2 \frac{4C^2}{l^4} = \frac{4C^2}{l^2}.$$

Hence the series  $\sum_{l \geq 2} \sum_{j=1}^{l^2} |\lambda^{jp_l} - 1|^2$  is convergent for all  $\lambda \in K$ , that is the series  $\sum_{k \geq 1} |\lambda^{n_k} - 1|^2$  is convergent for all  $\lambda \in K$ . We have thus proved the first part of our statement, namely that  $G_2((n_k))$  is uncountable.

Let now  $\sigma$  be a continuous probability measure on  $\mathbb{T}$ . The proof that  $\sigma$  cannot be IP-Dirichlet with respect to the sequence  $(n_k)_{k \geq 1}$  relies on the following lemma:

**Lemma 5.1.** — *For all  $l \geq 2$  and all  $s \geq 1$ ,  $sp_l$  belongs to the set*

$$\left\{ \sum_{k \in F} n_k ; F \in \mathcal{F}, \min(F) \geq M_{l-1} + 1 \right\}.$$

*Proof of Lemma 5.1.* — It is clear that for all  $n \geq 1$ ,

$$\left\{ \sum_{j \in F} j ; F \subseteq \{1, \dots, n\}, F \neq \emptyset \right\} = \left\{ 1, \dots, \frac{n(n+1)}{2} \right\}.$$

Hence

$$\left\{ \sum_{j \in F} jp_l ; F \subseteq \{1, \dots, l^2\}, F \neq \emptyset \right\} = \left\{ p_l, 2p_l, \dots, \frac{l^2(l^2+1)}{2} p_l \right\},$$

i.e.

$$\left\{ \sum_{k \in F} n_k ; F \subseteq \{M_{l-1} + 1, \dots, M_l\}, F \neq \emptyset \right\} = \{p_l, 2p_l, \dots, p_{l+1}\}.$$

This proves Lemma 5.1 for  $s \in \{1, \dots, \frac{l^2(l^2+1)}{2}\}$ . Then since

$$\left\{ \sum_{k \in F} n_k ; F \subseteq \{M_l + 1, \dots, M_{l+1}\}, F \neq \emptyset \right\} = \{p_{l+1}, 2p_{l+1}, \dots, \frac{(l+1)^2((l+1)^2+1)}{2} p_{l+1}\},$$

we get that

$$\begin{aligned} & \left\{ \sum_{k \in F} n_k ; F \subseteq \{M_{l-1} + 1, \dots, M_{l+1}\}, F \neq \emptyset \right\} \\ &= \{p_l, 2p_l, \dots, p_{l+1}, p_{l+1} + p_l, p_{l+1} + 2p_l, \dots, \\ & \quad 2p_{l+1}, \dots, \frac{(l+1)^2((l+1)^2+1)}{2} p_{l+1}\} \\ &= \{p_l, 2p_l, \dots, \frac{l^2(l^2+1)}{2} p_l, \frac{(l+1)^2((l+1)^2+1)}{2} p_l\}. \end{aligned}$$

Continuing in this fashion we obtain that for all  $q \geq 1$ ,

$$\begin{aligned} & \left\{ \sum_{k \in F} n_k \quad ; \quad F \subseteq \{M_{l-1} + 1, \dots, M_{l+q}\}, F \neq \emptyset \right\} \\ &= \{p_l, 2p_l, \dots, \prod_{j=0}^q \frac{(l+j)^2((l+j)^2+1)}{2} p_l\}. \end{aligned}$$

The conclusion of Lemma 5.1 follows from this.  $\square$

Suppose now that  $\sigma$  is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . Let  $l_0 \geq 2$  be such that for every  $F \in \mathcal{F}$  with  $\min(F) \geq M_{l_0-1} + 1$ ,  $|\hat{\sigma}(\sum_{k \in F} n_k)| \geq 1/2$ . Then Lemma 5.1 implies that for all  $s \geq 1$ ,  $|\hat{\sigma}(sp_{l_0})| \geq 1/2$ . This contradicts the continuity of the measure  $\sigma$ .

## 6. Additional results and comments

**6.1. A remark about the Erdős-Taylor sequence.** — Let  $(n_k)_{k \geq 1}$  be the sequence of integers defined by  $n_1 = 1$  and  $n_{k+1} = kn_k + 1$  for every  $k \geq 1$ . This sequence is interesting in our context because  $G_1((n_k)) = \{1\}$  while  $G_2((n_k))$  is uncountable ([5], see also [1]): if  $\lambda \in \mathbb{T} \setminus \{1\}$ , there exists a positive constant  $\varepsilon$  such that  $|\lambda^{n_k} - 1| \geq \frac{\varepsilon}{k}$  for all  $k \geq 1$ . Indeed, if for some  $k$  we have  $|\lambda^{n_k} - 1| \leq \frac{\varepsilon}{k}$ , then  $|\lambda^{kn_k} - 1| \leq \varepsilon$ , so that  $|\lambda^{n_{k+1}} - 1| \geq |\lambda - 1| - \varepsilon$ . This is a contradiction for  $\varepsilon = \frac{1}{2}|\lambda - 1|$ . Hence if  $\lambda \in \mathbb{T} \setminus \{1\}$  the series  $\sum_{k \geq 1} |\lambda^{n_k} - 1|$  is divergent. On the other hand, since the series  $\sum_{k \geq 1} (n_k/n_{k+1})^2$  is convergent,  $G_2((n_k))$  is uncountable. It is proved in [1] that there exists a continuous probability measure  $\sigma$  on  $\mathbb{T}$  which is IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . This statement can also be seen as a consequence of Theorem 2.2 of [8]: it is shown there that there exists a continuous generalized Riesz product  $\sigma$  on  $\mathbb{T}$  and a  $\delta > 0$  such that

$$|\hat{\sigma}(\sum_{k \in F} n_k)| \geq \delta$$

for every  $F \in \mathcal{F}$  such that  $\min(F) > 4$ . It is not difficult to see that this measure  $\sigma$  is in fact IP-Dirichlet with respect to  $(n_k)_{k \geq 1}$ . We briefly give the argument below. It can be generalized to all sequences  $(n_k)_{k \geq 1}$  such that the series  $\sum_{k \geq 1} (n_k/n_{k+1})^2$  is convergent, thus yielding another proof of Corollary 3.2.

The measure  $\sigma$  of [8] is constructed in the following way: let  $\Delta$  be the function defined for  $t \in \mathbb{R}$  by  $\Delta(t) = \max(1 - 6|t|, 0)$ . If  $K$  is the function  $\mathbb{R}$  given by the expression

$$K(t) = \frac{1}{2\pi} \left( \frac{\sin \frac{t}{2}}{\frac{t}{2}} \right)^2, \quad t \in \mathbb{R}$$

and  $K_\alpha$  is defined for each  $\alpha > 0$  by  $K_\alpha(t) = \alpha K(\alpha t)$ ,  $t \in \mathbb{R}$ , then  $\Delta(x) = \hat{K}_{\frac{1}{6}}(x)$  for every  $x \in \mathbb{R}$ . The function  $\Delta * \Delta$  is a  $\mathcal{C}^2$  function on  $\mathbb{R}$  which is supported on  $[-\frac{1}{3}, \frac{1}{3}]$ , takes positive values on  $] -\frac{1}{3}, \frac{1}{3}[$ , and attains its maximum at the point 0. Hence its derivative vanishes at the point 0. Let  $a > 0$  be such that the function  $\varphi = a\Delta * \Delta$  satisfies  $\varphi(0) = 1$ . We have also  $\varphi'(0) = 0$ , and so there exists a constant  $c \geq 0$  and a  $\gamma \in (0, \frac{1}{3})$  such that for all  $x$  with  $|x| < \gamma$ ,  $\varphi(x) \geq 1 - cx^2$ . Lastly, recall that  $\varphi(x) = \widehat{aK_{\frac{1}{6}}^2}(x)$  for all

$x \in \mathbb{R}$ . Consider now the sequence  $(P_j)_{j \geq 1}$  of trigonometric polynomials defined on  $\mathbb{T}$  in the following way: for  $j \geq 1$  and  $t \in \mathbb{R}$ ,

$$P_j(e^{it}) = \sum_{s \in \mathbb{Z}} \varphi\left(\frac{s}{j}\right) e^{ist}.$$

This is indeed a polynomial of degree at most  $\lfloor \frac{j}{3} \rfloor$ , since  $\varphi(\frac{s}{j}) = 0$  as soon as  $\frac{s}{j} \geq \frac{1}{3}$ . We now claim that  $P_j$  takes only nonnegative values on  $\mathbb{T}$ : indeed, consider for each  $j \geq 1$  and  $t \in \mathbb{R}$  the function  $\Phi_{j,t}$  defined by  $\Phi_{j,t}(x) = jK_{\frac{1}{6}}^2(j(x+t))$ ,  $x \in \mathbb{R}$ . Its Fourier transform is then given by  $\hat{\Phi}_{j,t}(\xi) = e^{i\xi t} \widehat{K_{\frac{1}{6}}^2}(\frac{\xi}{j}) = e^{i\xi t} \Delta * \Delta(\frac{\xi}{j})$ . Thus  $P_j(e^{it}) = a \sum_{s \in \mathbb{Z}} \hat{\Phi}_{j,t}(s)$ . Applying the Poisson formula to the function  $\Phi_{j,t}$ , we get that  $P_j(e^{it}) = 2\pi a \sum_{s \in \mathbb{Z}} \Phi_{j,t}(2\pi s) = 2\pi a \sum_{s \in \mathbb{Z}} jK_{\frac{1}{6}}^2(j(2\pi s+t)) \geq 0$ . Hence  $P_j(e^{it})$  is nonnegative for all  $t \in \mathbb{R}$ ,  $\hat{P}_j(0) = 1$  and  $\hat{P}_j(1) = \varphi(\frac{1}{j}) \geq 1 - \frac{c}{j^2}$  as soon as  $j \geq j_0$ , where  $j_0 = \lfloor \frac{1}{\gamma} \rfloor + 1$ . Consider then for  $m \geq j_0$  the nonnegative polynomials  $Q_m$  defined by

$$Q_m(e^{it}) = \prod_{j=j_0}^m P_j(e^{in_j t}), \quad t \in \mathbb{R}.$$

We have  $\hat{Q}_m(0) = 1$  for each  $m \geq 1$ . Moreover, since the degree of  $P_j$  is less than  $\lfloor \frac{j}{3} \rfloor$  and  $n_{j+1} > \frac{jn_j}{3}$ , the polynomials  $Q_m$  converge in the  $w^*$ -topology to a generalized Riesz product  $\sigma$  on  $\mathbb{T}$  which is continuous and such that for every set  $F \in \mathcal{F}$  with  $\min(F) \geq j_0$ ,

$$\hat{\sigma}\left(\sum_{k \in F} n_k\right) \geq \prod_{k \in F} \left(1 - \frac{c}{k^2}\right).$$

It follows that  $\sigma$  is an IP-Dirichlet measure with respect to the sequence  $(n_k)_{k \geq 1}$ .

**6.2. A sequence  $(n_k)_{k \geq 1}$  with respect to which there exists a continuous Dirichlet measure, but such that  $G_\infty((n_k)) = \{1\}$ .** — The examples of sequences  $(n_k)_{k \geq 1}$  given in [3] and [4] for which there exists a continuous probability measure  $\sigma$  on  $\mathbb{T}$  such that  $\hat{\sigma}(n_k) \rightarrow 1$  as  $k \rightarrow +\infty$  all share the property that  $|\lambda^{n_k} - 1| \rightarrow 0$  for some  $\lambda \in \mathbb{T} \setminus \{1\}$ . One may thus wonder whether there exists a sequence  $(n_k)_{k \geq 1}$  with respect to which there exists a continuous Dirichlet probability measure  $\sigma$ , and such that  $G_\infty((n_k)) = \{\lambda \in \mathbb{T} ; |\lambda^{n_k} - 1| \rightarrow 0\} = \{1\}$ . The answer is yes, and an ad hoc sequence  $(n_k)_{k \geq 1}$  can be constructed from the Erdős-Taylor sequence above. Changing notations, let us denote by  $(p_k)_{k \geq 1}$  this sequence defined by  $p_1 = 1$  and  $p_{k+1} = kp_k + 1$  for each  $k \geq 1$ . For each integer  $q \geq 1$ , consider the finite set

$$\mathcal{P}_q = \left\{ \sum_{k \in F} p_k ; F \neq \emptyset, F \subseteq \{2^q + 1, \dots, 2^{q+1}\} \right\}.$$

The set  $\bigcup_{q \geq 1} \mathcal{P}_q$  can be written as  $\{n_k ; k \geq 1\}$ , where  $(n_k)_{k \geq 1}$  is a strictly increasing sequence of integers. Let now  $\sigma$  be a continuous probability measure which is IP-Dirichlet with respect to the Erdős-Taylor sequence  $(p_k)_{k \geq 1}$ :

$$\hat{\sigma}\left(\sum_{k \in F} p_k\right) \rightarrow 1 \quad \text{as} \quad \min(F) \rightarrow +\infty, F \in \mathcal{F}.$$

This implies that  $\hat{\sigma}(n_k) \rightarrow 1$  as  $k \rightarrow +\infty$ . Indeed, let  $\varepsilon > 0$  and  $k_0$  be such that  $|\hat{\sigma}(\sum_{k \in F} p_k) - 1| < \varepsilon$  for all  $F \in \mathcal{F}$  with  $\min(F) \geq k_0$ . Let  $q_0$  be such that  $2^{q_0} + 1 \geq k_0$ . Then  $|\hat{\sigma}(n_k) - 1| < \varepsilon$  for all  $k$  such that  $n_k$  belongs to the union  $\bigcup_{q \geq q_0} \mathcal{P}_q$ . Since all the sets  $\mathcal{P}_q$  are finite,  $|\hat{\sigma}(n_k) - 1| < \varepsilon$  for all but finitely many  $k$ .

It remains to prove that  $G_\infty((n_k)) = \{1\}$ , and the argument for this is very close to one employed in [1]. Let  $\varepsilon \in (0, 1/16)$  for instance, and suppose that  $\lambda \in \mathbb{T}$  is such that  $|\lambda^{n_k} - 1| < \varepsilon$  for all  $k$  larger than some  $k_0$ . We claim then that if  $q_0$  is such that  $2^{q_0} + 1 \geq k_0$ , then we have for all  $q$  larger than  $q_0$

$$(13) \quad \sum_{k=2^q+1}^{2^{q+1}} |\lambda^{p_k} - 1| < 2C^2\varepsilon,$$

where  $C > 0$  is a constant such that  $\{t\}/C \leq |e^{2i\pi t} - 1| \leq C\{t\}$  for all  $t \in \mathbb{R}$ . Indeed, our assumption that  $|\lambda^{n_k} - 1| < \varepsilon$  for all  $k \geq k_0$  implies that for all  $q \geq q_0$  and all disjoint finite subsets  $F$  and  $G$  of the set  $\mathcal{P}_q$ ,

$$\left\{ \sum_{k \in F} p_k \theta \right\} < C\varepsilon, \quad \left\{ \sum_{k \in G} p_k \theta \right\} < C\varepsilon \quad \text{and} \quad \left\{ \sum_{k \in F \sqcup G} p_k \theta \right\} < C\varepsilon$$

where  $\lambda = e^{2i\pi\theta}$  with  $\theta \in [0, 1)$  and  $F \sqcup G$  denotes the disjoint union of  $F$  and  $G$ . Now the same argument as in [1, Prop. 1.1] yields that

$$\left\langle \sum_{k \in F \sqcup G} p_k \theta \right\rangle = \left\langle \sum_{k \in F} p_k \theta \right\rangle + \left\langle \sum_{k \in G} p_k \theta \right\rangle.$$

Setting

$$A_{q,+} = \{k \in \{2^q + 1, \dots, 2^{q+1}\} ; \langle p_k \theta \rangle \geq 0\}$$

and

$$A_{q,-} = \{k \in \{2^q + 1, \dots, 2^{q+1}\} ; \langle p_k \theta \rangle < 0\},$$

this implies that

$$\sum_{k \in A_{q,+}} \{p_k \theta\} < C\varepsilon \quad \text{and} \quad \sum_{k \in A_{q,-}} \{p_k \theta\} < C\varepsilon.$$

Hence

$$\sum_{k=2^q+1}^{2^{q+1}} \{p_k \theta\} < 2C\varepsilon \quad \text{so that} \quad \sum_{k=2^q+1}^{2^{q+1}} |\lambda^{p_k} - 1| < 2C^2\varepsilon \quad \text{for all } q \geq q_0.$$

Suppose now that  $\lambda \neq 1$ , and set  $\varepsilon = |\lambda - 1|/(4C^2)$ . Then (13) above implies that there exists an infinite subset  $E$  of  $\mathbb{N}$  such that  $|\lambda^{p_k} - 1| \leq (2C^2\varepsilon)/k$  for all  $k \in E$ . If it were not the case, we would have  $|\lambda^{p_k} - 1| > (2C^2\varepsilon)/k$  for all  $k$  large enough, so that

$$(14) \quad \sum_{k=2^q+1}^{2^{q+1}} |\lambda^{p_k} - 1| > 2C^2\varepsilon \sum_{k=2^q+1}^{2^{q+1}} \frac{1}{k} \geq 2C^2\varepsilon \frac{2^{q+1} - 2^q}{2^q} \geq 2C^2\varepsilon$$

for all  $q$  large enough, which is a contradiction with (13). This proves the existence of the set  $E$ . Now for all  $k \in E$

$$|\lambda^{p_{k+1}} - 1| \geq |\lambda - 1| - |\lambda^{p_k} - 1| \geq |\lambda - 1| - k|\lambda^{p_k} - 1| \geq 4C^2\varepsilon - 2C^2\varepsilon = 2C^2\varepsilon.$$

But this stands again in contradiction with (13), and we infer from this that  $\lambda$  is necessarily equal to 1. Thus  $G_\infty((n_k)) = \{1\}$ , and we are done.

## References

- [1] J. AARONSON, M. HOSSEINI, M. LEMAŃCZYK, IP-rigidity and eigenvalue groups, preprint 2012, arxiv 1203.2257.
- [2] C. BADEA, S. GRIVAUX, Unimodular eigenvalues, uniformly distributed sequences and linear dynamics, *Adv. Math.* **211** (2007), pp 766 – 793.
- [3] V. BERGELSON, A. DEL JUNCO, M. LEMAŃCZYK, J. ROSENBLATT, Rigidity and non-recurrence along sequences, preprint 2011, arxiv 1103.0905.
- [4] T. EISNER, S. GRIVAUX, Hilbertian Jamison sequences and rigid dynamical systems, *J. Funct. Anal.* **261** (2011), pp 302 – 329.
- [5] P. ERDÖS, S. TAYLOR, On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences, *Proc. London Math. Soc.* **7** (1957), pp 598 – 615.
- [6] B. HOST, J. -F. MÉLA, F. PARREAU, Analyse harmonique des mesures (Séminaire de Villetaneuse), *Astérisque* **135/136** (1986), Société Mathématique de France.
- [7] B. HOST, J. -F. MÉLA, F. PARREAU, Non singular transformations and spectral analysis of measures, *Bull. Soc. Math. Fr.* **119** (1991), pp 33 – 90.
- [8] Y. KATZNELSON, Sequences of integers dense in the Bohr group, *Proc. Roy. Inst. of Tech. Stockholm* (1973), pp 79 – 86.
- [9] F. PARREAU, Ergodicité et pureté des produits de Riesz, *Ann. Inst. Fourier* **40** (1990), pp 391 – 405.
- [10] M. QUEFFÉLEC, Substitution dynamical systems – spectral analysis, second edition, *Lecture Notes in Mathematics* **1294**, Springer-Verlag, Berlin, 2010.

---

September 14, 2012

SOPHIE GRIVAUX, CNRS, Laboratoire Paul Painlevé, UMR 8524, Université Lille 1, Cité Scientifique,  
59655 Villeneuve d'Ascq Cedex, France • E-mail : [grivaux@math.univ-lille1.fr](mailto:grivaux@math.univ-lille1.fr)